Realization of compact Lie algebras in Kähler manifolds

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Abstract. The Berezin quantization on a simply connected homogeneous Kähler manifold, which is considered as a phase space for a dynamical system, enables a description of the quantal system in a (finite-dimensional) Hilbert space of holomorphic functions corresponding to generalized coherent states. The Lie algebra associated with the manifold symmetry group is given in terms of first-order differential operators. In the classical theory, the Lie algebra is represented by the momentum maps which are functions on the manifold, and the Lie product is the Poisson bracket given by the Kähler structure. The Kähler potentials are constructed for the manifolds related to all compact semi-simple Lie groups. The complex coordinates are introduced by means of the Borel method. The Kähler structure is obtained explicitly for any unitary group representation. The cocycle functions for the Lie algebra and the Killing vector fields on the manifold are also obtained.

1. Introduction

Action of any Lie group on its homogeneous spaces is determined by realization of the corresponding Lie algebra by means of first-order differential operators and the Killing vector fields. Under some conditions, the group representations can be constructed in terms of linear operators in a Hilbert space of functions on the manifold. In general, this construction is called *geometric quantization*, and the Hilbert space may be considered as a space of states of a quantum dynamical system. Moreover, the manifold is the phase space of a classical dynamical system with an appropriate symplectic structure and Poisson brackets, representing the Lie algebra of the transformation group. An important class of such homogeneous manifolds, provided with a complex Kähler structure, has been constructed on the basis of the Borel theory [1]. A general theory of quantization on Kähler manifolds was developed by Berezin [2–4]. In particular, examples of homogeneous manifolds were given in his works. Another approach to geometric quantization is based upon the method of coadjoint orbits proposed by Kirillov [5] and Kostant [6]. (Reviews may be found in [7–9].) The simplest example, where the group is $SU(2)$ and the manifold is S^2, was given previously by Souriau [10].

The purpose of this work is an explicit construction of the Berezin quantization for compact Lie groups. Any unitary group representation corresponds to a (finite, for compact groups) Hilbert space of holomorphic functions. The system states can be considered also as *generalized coherent states*, introduced by Perelomov [11]. Section 2 is a short review of the Berezin quantization, i.e. construction of the Hilbert space of holomorphic functions on Kähler manifolds. Section 3 is an exposition of two realizations of the Lie algebra on homogeneous Kähler manifolds: first, in terms of commutators of holomorphic differential operators, and second, in terms of suitably defined Poisson brackets. In section 4, the

Borel parametrization is introduced for homogeneous manifolds, and the Kähler potentials are constructed, using the projection matrices of [12]. Explicit expressions for operator symbols and momentum maps in the local coordinates are presented, which is the main result of this work. The subject of section 5 is a construction of the differential operators representing the Lie algebra in the space of holomorphic functions on the manifold. The *appendix contains (i) the derivation of an explicit expression for the Kähler potentials and* the momentum maps, and (ii) an example: the homogeneous manifolds for unitary groups.

The notation follows standard texts, e.g. Kobayashi and Nomizu [13]. The sum over repeated indices is implied throughout the paper.

2. Berezin quantization on compact Kähler manifolds

Let $\mathcal{M}$ be a Kähler manifold, and $K(z, \bar{z})$ be the Kähler potential, defined in any open coordinate neighbourhood of the manifold with local complex coordinates z_α ($\alpha = 1, \ldots, m \equiv \dim \mathcal{M}$). If a group $\mathcal{G}$ is acting holomorphically on $\mathcal{M}$, $z \to gz, \forall g \in \mathcal{G}$, the Kähler potential is transformed according to

$$K(gz, \overline{gz}) = K(z, \bar{z}) + \Phi(z; g) + \overline{\Phi(z; g)} \tag{1}$$

where $\Phi(z; g)$ is locally a holomorphic function of z. This function must satisfy the cocycle condition

$$\Phi(z; g_2 g_1) = \Phi(g_1 z; g_2) + \Phi(z; g_1) \qquad \forall g_1, g_2 \in \mathcal{G} \tag{2}$$

which results from the group property $z \to g_2(g_1 z) \equiv (g_2 g_1)z$.

The corresponding Kähler (1,1)-form has the following coordinate representation

$$\omega \equiv \omega_{\alpha\bar{\beta}}(z, \bar{z}) \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\beta \qquad \omega_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K \tag{3}$$

where $\partial_\alpha \equiv \partial/\partial z^\alpha$, $\partial_{\bar{\beta}} \equiv \partial/\partial\bar{z}^\beta$. The form $\omega = -\bar{\omega}$ is closed, $\partial\omega = 0$, $\bar{\partial}\omega = 0$, and invariant under the group transformations, as follows from (1). The form is called *integral* if, being integrated over any two-dimensional cycle in $\mathcal{M}$, it gives an integer, times $2\pi i$. The corresponding Kähler potential is also called integral.

Any integral Kähler potential is associated with a holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}$. The holomorphic sections of $\mathcal{L}$ will be denoted by ψ; they represent states of a quantum mechanical system and are given by locally holomorphic *wavefunctions* $\psi(z)$. Following Berezin [3], a Hilbert space structure is assigned to $\mathcal{L}$ by means of the following $\mathcal{G}$-invariant scalar product

$$(\psi_2, \psi_1) = C \int_{\mathcal{M}} \overline{\psi_2(z)}\psi_1(z) \exp[-K(z, \bar{z})] \, \mathrm{d}\mu(z, \bar{z}) \,. \tag{4}$$

Here the invariant volume element is the mth power of ω, $\mathrm{d}\mu = i\omega \wedge \cdots \wedge i\omega$ (m times), and C is a normalization constant (see equation (10) below). (We assume that the form ω is *non-degenerate*, so the total volume is non-zero.) Under the action of the group, the wavefunctions are subject to transformations, corresponding to (1), i.e.

$$(\hat{U}(g^{-1})\psi)(z) = \exp[-\Phi(z; g)]\psi(gz) \tag{5}$$

where $\hat{U}(g)$ is the operator representing the group element g in the Hilbert space, $\hat{U}(g_1)\hat{U}(g_2) = \hat{U}(g_1 g_2)$.

Operators in the Hilbert space, which describe observables of the quantal system, are represented with their symbols, e.g. $\hat{A} \to A(z, \bar{\zeta})$, which are mappings $\mathcal{M} \times \mathcal{M} \to \mathbb{C}^\times$. The action of the operators in the Hilbert space is given in terms of their symbols [3]

$$(\hat{A}\psi)(\zeta) = C \int_{\mathcal{M}} A(\zeta, \bar{z})\psi(z) \exp[K(\zeta, \bar{z}) - K(z, \bar{z})] \, \mathrm{d}\mu(z, \bar{z}) \,. \tag{6}$$

Berezin proved, in particular, that any element of the Hilbert space is reproduced by the integral with a unit symbol

$$\psi(\zeta) \equiv C \int_{\mathcal{M}} \psi(z) \exp\left[K(\zeta, \bar{z}) - K(z, \bar{z})\right] d\mu(z, \bar{z}) \qquad \forall \psi \in \mathcal{L}. \qquad (7)$$

Moreover, the trace of any operator is given by the integral of its symbol

$$\mathrm{tr}(\hat{A}) = C \int_{\mathcal{M}} A(z, \bar{z}) \, d\mu(z, \bar{z}). \qquad (8)$$

For compact manifolds, the trace of the identity operator $\hat{I}$ exists and equals the total number of states N, which is finite in this case

$$\hat{I} \rightarrow I(\zeta, \bar{z}) \equiv 1 \qquad \mathrm{tr}(\hat{I}) \equiv N = C \int_{\mathcal{M}} d\mu(z, \bar{z}). \qquad (9)$$

Constant sections of $\mathcal{L}$ belong to the Hilbert space for the compact case, and in order to fix the normalization we assume that $\psi(z) \equiv 1$ has the unit norm. As a result, the constant C is expressed in terms of the volume V of the manifold

$$C^{-1} = \int_{\mathcal{M}} \exp[-K(z, \bar{z})] \, d\mu(z, \bar{z}) = V/N \qquad V \equiv \int_{\mathcal{M}} d\mu(z, \bar{z}). \qquad (10)$$

The Hermitian conjugation in the Hilbert space is represented by the complex conjugation of the symbol and transposition of its arguments

$$\hat{A}^\dagger \rightarrow A^*(\zeta, \bar{z}) = \overline{A(z, \bar{\zeta})}. \qquad (11)$$

As follows from (6), the symbol for product of operators is given by an integral of product of their symbols (the $*$-product)

$$\hat{A}\hat{B} \rightarrow (A * B)(\zeta, \bar{\eta}) \equiv \frac{N}{V} \int_{\mathcal{M}} A(\zeta, \bar{z}) B(z, \bar{\eta}) \exp\Big[K(\zeta, \bar{z}) - K(z, \bar{z})$$

$$+ K(z, \bar{\eta}) - K(\zeta, \bar{\eta})\Big] d\mu(z, \bar{z}). \qquad (12)$$

Thus the associative algebra of observables for the quantal system is constructed completely in terms of the operator symbols. The symbols are functions of two independent variables, holomorphic in the first, and anti-holomorphic in the second. The 'physical observables' are the reductions to the phase space, where $\zeta = z$.

3. The symbol representation of the Lie algebra

3.1. Differential operators

As soon as the transformation law for the wavefunctions is given by (5), the symbols of the operators representing the group elements are given in terms of the Kähler potential and the transition functions of (1)

$$(\hat{U}(g)\psi)(\zeta) = \frac{N}{V} \int_{\mathcal{M}} U_g(\zeta, \bar{z}) \psi(z) \exp\left[K(\zeta, \bar{z}) - K(z, \bar{z})\right] d\mu(z, \bar{z}) \qquad (13)$$

where

$$U_g(\zeta, \bar{z}) = \exp\left[K(g^{-1}\zeta, \bar{z}) - K(\zeta, \bar{z}) - \Phi(\zeta, g^{-1})\right]$$

$$\equiv \exp\left[K(\zeta, \overline{gz}) - K(\zeta, \bar{z}) - \overline{\Phi(z, g)}\right]. \qquad (14)$$

The second equality follows immediately from (1); it is actually equivalent to the unitarity condition, $U_{g^{-1}}(\zeta, \bar{z}) = \overline{U_g(z, \bar{\zeta})}$. The symbol of the group element is subject to the following transformation law

$$\hat{U}(h^{-1})\hat{U}(g)\hat{U}(h) \to U_{h^{-1}gh}(\zeta, \bar{z}) \equiv U_g(h\zeta, \overline{hz}) \qquad \forall h \in \mathcal{G}. \tag{15}$$

Let us turn to the Lie algebra **g** of the group $\mathcal{G}$. Introducing a basis τ_a in **g**, with $a = 1, \ldots, n \equiv \dim \mathbf{g}$, one has a Cartesian coordinate representation of the group elements and the corresponding Lie derivatives D_a in the group manifold

$$g = \exp(-\xi^a \tau_a) \qquad D(\xi)_a \equiv -L_a^b(\xi)\partial/\partial \xi^b \tag{16}$$

where ξ^a are the group parameters, and the field L_a^b is given in terms of the adjoint group representation, so that $g^{-1} D_a g = \tau_a$. The action of the group $\mathcal{G}$ in the manifold determines the holomorphic Killing fields

$$\nabla_a = \kappa_a^\alpha(z)\partial_\alpha \qquad \kappa_a^\alpha(z) \equiv D(\xi)_a(gz)^\alpha|_{g=e} \tag{17}$$

(e is the unit group element). The conjugate Killing fields and the differential operators $\overline{\nabla}_a$ are defined similarly. Now the differential operators $\hat{T}_a$, which act upon elements of the Hilbert space $\mathcal{L}$ and represent the basis in **g**, are obtained from (5)

$$\tau_a \to \hat{T}_a = \nabla_a - \varphi_a(z) \qquad \varphi_a(z) = D(\xi)_a \Phi(z; g)|_{g=e}. \tag{18}$$

The Lie multiplication is given by that for the basis elements

$$[\tau_a, \tau_b] = f_{ab}^c \tau_c \quad \to \quad [\hat{T}_a, \hat{T}_b] = f_{ab}^c \hat{T}_c. \tag{19}$$

One can introduce a real basis, where the *structure constants* f_{ab}^c are real, and $\hat{T}_a$ are anti-self-adjoint operators with respect to the scalar product (4). The same relations are satisfied by the differential operators D_a and ∇_a, while $\varphi_a(z)$ satisfy the linear differential equations

$$\nabla_a \varphi_b - \nabla_b \varphi_a = f_{ab}^c \varphi_c. \tag{20}$$

The solution to these equations is given in section 5.

3.2. Symbols

Elements of the Lie algebra are also represented by their symbols, which can be obtained from symbols of the group elements, given in (14)

$$\hat{T}_a \to T_a(\zeta, \bar{z}) = -\overline{T_a(z, \bar{\zeta})}$$

$$= D_a U_g(\zeta, \bar{z})|_{g=e}$$

$$= \nabla_a K(\zeta, \bar{z}) - \varphi_a(\zeta)$$

$$\equiv -\overline{\nabla}_a K(\zeta, \bar{z}) + \overline{\varphi_a(z)}. \tag{21}$$

(The first equality is true for the real basis, as soon as $\hat{T}_a = -\hat{T}_a^\dagger$; the last one follows directly from (1).) Thus the symbols of the Lie algebra are expressed in terms of the coefficients of the symplectic one-form generated by the Kähler structure

$$T_a(\zeta, \bar{z}) = \kappa_a^\alpha(\zeta)\Lambda_\alpha(\zeta, \bar{z}) - \varphi_a(\zeta) \tag{22}$$

where Λ_α are coefficients of the one-form

$$\mathrm{d}K = \Lambda_\alpha \mathrm{d}\zeta^\alpha + \overline{\Lambda_\beta}\mathrm{d}\bar{z}^\beta. \tag{23}$$

Evidently, the symbols satisfy the following relations:

$$\partial_\alpha T_a(z,\bar{z}) = \omega_{\alpha\bar{\beta}}\overline{\kappa_a^\beta}(z) \qquad \partial_{\bar{\beta}}T_a(z,\bar{z}) = -\omega_{\alpha\bar{\beta}}\kappa_a^\alpha(z). \qquad (24)$$

The functions $T_a(z,\bar{z})$ are called (equivariant) *momentum maps*. It can be seen from (15) that their transformation law is given by the adjoint group representation $g \to A(g)$

$$g^{-1}\tau_a g = A_a^b(g)\tau_b \to T_a(g\zeta, \overline{g\bar{z}}) = A_a^b(g)T_b(\zeta, \bar{z}) \qquad \forall g \in \mathcal{G}. \qquad (25)$$

Applying the Lie equations for ∇_a and (20), from (21) and (24) one obtains

$$-\omega_{\alpha\bar{\beta}}(\kappa_a^\alpha\overline{\kappa_b^\beta} - \kappa_b^\alpha\overline{\kappa_a^\beta}) = \nabla_a T_b - \nabla_b T_a = f_{ab}^c T_c. \qquad (26)$$

This equality is equivalent to the fundamental property of the momentum maps: they implement a realization of the Lie algebra g in terms of the Poisson brackets

$$\{T_a, T_b\}_{\text{P.B.}} = f_{ab}^c T_c. \qquad (27)$$

We have introduced here the Poisson brackets in $\mathcal{M}$, which are determined by a field ϖ dual to the form ω. Namely, for any two symbols $A(z,\bar{z})$ and $B(z,\bar{z})$ the definition is

$$\{A, B\}_{\text{PB}} \equiv \varpi^{\alpha\bar{\beta}}(\partial_\alpha A \partial_{\bar{\beta}}B - \partial_\alpha B \partial_{\bar{\beta}}A) = -\{B, A\}_{\text{PB}}$$

$$\omega_{\alpha\bar{\beta}}\varpi^{\alpha\bar{\gamma}} = \delta_{\bar{\beta}}^{\bar{\gamma}} \qquad \omega_{\alpha\bar{\beta}}\varpi^{\gamma\bar{\beta}} = \delta_\alpha^\gamma. \qquad (28)$$

With this definition, $\overline{\{A, B\}}_{\text{PB}} = \{\bar{B}, \bar{A}\}_{\text{PB}}$, and the Jacobi identity for the Poisson brackets results from the fact that ω defined in (3) is a closed form.

The equivalence of the commutators (19) and the Poisson brackets (27) is the reason why the operator–symbol correspondence is called *quantization*. One should note that the commutator–Poisson bracket correspondence holds only for elements of the Lie algebra g, but not necessarily for any pair of observables, which belong to the enveloping algebra.

The construction of the Kähler potentials for the homogeneous manifolds, described in the next section, enables an explicit derivation of the momentum maps.

4. The Kähler structure on compact homogeneous manifolds

4.1. Flag manifolds

Let $\mathcal{G}$ be a compact simple Lie group, and $\mathcal{T}$ be its maximal Abelian subgroup (the maximal torus). The coset space $\mathcal{F} = \mathcal{G}/\mathcal{T}$ (the flag manifold) is provided with a Kähler structure [1]. We shall introduce a local complex parametrization in $\mathcal{F}$ by means of the canonical diffeomorphism $\mathcal{G}/\mathcal{T} \cong \mathcal{G}^c/\mathcal{P}$. Here $\mathcal{G}^c$ is the complexification of $\mathcal{G}$; the parabolic subgroup $\mathcal{P}$ satisfies the requirements $\mathcal{P} \supset \mathcal{B}$, where $\mathcal{B}$ is a Borel subgroup of $\mathcal{G}$ and $\mathcal{P} \cap \mathcal{G} = \mathcal{T}$.

We shall employ the canonical basis $\{\tau_a\} = \{h_j, e_{\pm\alpha}\}$ in the Lie algebra g, where $j = 1, \dots, r \equiv \text{rank}(\text{g})$ and $\{\alpha\} \in \Delta_\text{g}^+$ are the positive roots of g. In particular, the Lie products of the basis elements are

$$[h_j, h_k] = 0 \qquad [h_j, e_\alpha] = (\alpha \cdot w_j)e_\alpha. \qquad (29)$$

Here the $\{w_j\}$ are the fundamental weight vectors which constitute a system dual to primitive roots $\{\gamma^{(j)}\}$ in the root space, i.e.

$$2(\gamma^{(j)} \cdot w_{j'})/(\gamma^{(j)} \cdot \gamma^{(j)}) = \delta_{jj'}. \qquad (30)$$

(Any positive root α is a sum of the primitive roots with non-negative integer coefficients.) For any unitary irreducible group representation R_l, its highest weight l is given by a sum of the fundamental weights with non-negative integer coefficients

$$l = \sum_{j=l}^{r} l^j w_j \qquad l^j = 2(\gamma^{(j)} \cdot l)/(\gamma^{(j)} \cdot \gamma^{(j)}). \tag{31}$$

Given the canonical basis, the Lie algebra $\mathbf{g}$ is split into three subalgebras, $\mathbf{g} = \mathbf{g}^- \oplus \mathbf{t} \oplus \mathbf{g}^+$, corresponding to three subsets of the basis elements, $\{e_{-\alpha}\}$, $\{h_j\}$, $\{e_\alpha\}$. Respectively, the Lie algebra $\mathbf{g}^+$ generates a nilpotent subgroup $\mathcal{G}^+ \subset \mathcal{G}^c$, and the Lie algebra $\mathbf{p} = \mathbf{g}^- \oplus \mathbf{t}^c$ generates $\mathcal{P}$. Any element $g \in \mathcal{G}^c$ has a unique Mackey decomposition, $g = fp$, where $p \in \mathcal{P}$ and $f \in \mathcal{G}^+$ (the decomposition is valid for all g, except for a subset of a lower dimensionality). The complex parameters which can be introduced in $\mathcal{F}$ correspond to the positive roots of $\mathbf{g}$

$$f(z) = \exp\left(\sum_{\alpha \in \Delta_+^\bullet} z^\alpha e_\alpha \right) \qquad z^\alpha \in \mathbb{C}. \tag{32}$$

As soon as $f(z)$ is an element of a nilpotent group, its matrix representations are polynomials of z^α. The local form (32) for f is valid in a neighbourhood of the point $z^\alpha = 0$, i.e. the origin of the coordinate system in $\mathcal{F}$. Of course, the origin is not a special point, since the manifold is homogeneous; it is just related to the choice of coordinates. Transition to other domains of $\mathcal{F}$, covering the manifold completely, can be performed by means of the group transformations.

The group $\mathcal{G}^c$ acts on $\mathcal{F}$ by left multiplications. Actually, for any element g one has a unique decomposition

$$gf(z) = f(gz)p(z; g) \qquad p(z; g) \in \mathcal{P} \tag{33}$$

and gz is a *rational* function of z. For any element g which does not drive the point with coordinates z^α outside the coordinate neighbourhood containing the origin where (32) is valid, gz and $p(z; g)$ can be obtained from (33) by means of a linear algebra. Performing two consecutive transformations, as in (2), one obtains

$$p(z; g_2g_1) = p(g_1z; g_2)p(z, g_1) \qquad \forall g_1, g_2 \in \mathcal{G}^c. \tag{34}$$

The Lie equation for the holomorphic one-cocycles is derived from this cocycle condition in section 5. In view of (29), the decomposition (33) shows that $\mathcal{T}$ is a little group of $\mathcal{F}$.

4.2. Fundamental Kähler potentials

The solution to (1) was found by Bando, Kuratomo, Maskawa and Uehara [12] (see also [14]). Partial solutions are given in terms of projection matrices η_j, which exist in any matrix representation of $\mathcal{G}^c$ and correspond to elements of the Cartan subalgebra $h_j \in \mathbf{g}$. Their basic properties are as follows:

$$\eta_j = \eta_j^\dagger \qquad \eta_j^2 = \eta_j \qquad \eta_j \hat{h}_k = \hat{h}_k \eta_j \quad \forall j, k = 1, \ldots, r$$

$$\eta_j \hat{e}_{-\alpha} \eta_j = \hat{e}_{-\alpha} \eta_j \qquad \eta_j \hat{e}_\alpha \eta_j = \eta_i \hat{e}_\alpha. \tag{35}$$

(The hat stands for the matrix representation.) All η_j are commuting with each other. Respectively, in the group representation one has

$$\eta_j \hat{f} \eta_j = \eta_j \hat{f} \quad \forall f \in \mathcal{G}^+ \qquad \eta_j \hat{p} \eta_j = \hat{p} \eta_j \quad \forall p \in \mathcal{P}. \tag{36}$$

For any representation of $\mathcal{G}^c$, where $\hat{h}_j$ are diagonal, all η_j are also diagonal. Each η_j has 1 on the diagonal where $i\hat{h}_j$ has its minimum eigenvalues; all the other elements are 0. As is shown in [15], η_j can be expressed in terms of elements of $\mathcal{G}^c$. For any given g, the matrices η_j have different ranks for different j.

Given η, the *projected determinant* is defined for any matrix M as

$$\det_\eta M \equiv \det(\eta M \eta + I - \eta). \tag{37}$$

The operation is designed to be multiplicative for the Mackey decomposition, as follows from (36). Now the *fundamental* Kähler potentials and the corresponding transition functions are defined for any j, namely [12]

$$K^j(z, \bar{z}) \equiv \log \det_\eta \left(\hat{f}(z)^\dagger \hat{f}(z) \right) \tag{38}$$

$$\Phi^j(z; g) = - \log \det_\eta \hat{p}(z, g). \tag{39}$$

The transformation property (1) follows immediately from (33) and (36). The *fundamental* representation of $\mathcal{G}^c$ should be used for this construction. One can show that the use of the fundamental representation is essential in order to get *all* integral Kähler two-forms from the basic potentials (38). As soon as $\hat{f}(z)$ is a polynomial, $\exp[K^j(\zeta, \bar{z})]$ is also a polynomial in both sets of its arguments. As we show in the appendix, the fundamental Kähler potentials can be represented also in another, sometimes more suitable form

$$K^j(\zeta, \bar{z}) = \log \det{}'\left(\hat{f}(\zeta)\eta_j \hat{f}(z)^\dagger \right) \tag{40}$$

where $\det' M$ is the notation we use for product of all *non-zero* eigenvalues of M. (The projection matrices η are singular.) Similarly, the transition functions are given by

$$\Phi^j(z; g) = - \log \det{}'\left(\hat{g}\hat{f}(z)\eta_j \hat{f}(gz)^{-1} \right). \tag{41}$$

Now for any given unitary group representation specified with a dominant weight l, equation (31), one obtains the general expressions for the corresponding Kähler potential and the transition function

$$K^{(l)}(\zeta, \bar{z}) = \sum_{j=1}^r l_j K^j(\zeta, \bar{z}), \tag{42}$$

$$\Phi^{(l)}(z; g) = \sum_{j=1}^r l_j \Phi^j(z; g). \tag{43}$$

Respectively, $\exp K^{(l)}$ is a polynomial, and its degree is determined by l. The standard orthogonalization procedure leads to construction of a (finite) polynomial basis $\{\phi_\nu(z)\}$, starting from $\varphi_1 = 1$

$$\exp[K^{(l)}(\zeta, \bar{z})] = \sum_{\nu=1}^N \phi_\nu(\zeta)\overline{\phi_\nu(z)}. \tag{44}$$

Having the fundamental Kähler potentials, one obtains the desired representation of the Lie algebra.

4.3. Construction of the momentum maps

The explicit expressions for the Kähler potential (equations (38)–(41)) enable one to get a compact form for the symbol of the group element in (14). As follows from (33):

$$U_g^{(l)}(\zeta, \bar{z}) = \prod_{j=1}^{r} \left[\frac{\det_{\eta_j}(\hat{f}(z)^\dagger \hat{g}^{-1} \hat{f}(\zeta))}{\det_{\eta_j}(\hat{f}(z)^\dagger \hat{f}(\zeta))} \right]^{l_j} \equiv \prod_{j=1}^{r} \left[\frac{\det'(\hat{g}^{-1} \hat{f}(\zeta) \eta_j \hat{f}(z)^\dagger)}{\det'(\hat{f}(\zeta) \eta_j \hat{f}(z)^\dagger)} \right]^{l_j}. \tag{45}$$

In the local coordinates, the symbol of any group element is a rational function, its numerator being a polynomial with coefficients determined by the group element, while the denominator can be always chosen as the reproducing kernel (44).

In order to get the symbols for the basis elements of the Lie algebra g, $\tau_a \to T_a^{(l)}(\zeta, \bar{z})$, one has to apply the Lie derivative (equation (16)). The result is

$$T_a^{(l)}(\zeta, \bar{z}) = -\text{tr}\left(\rho^{(l)}(\zeta, \bar{z}) \hat{\tau}_a\right). \tag{46}$$

Here the trace is taken in the fundamental representation, and the symbols depend on l linearly

$$\rho^{(l)}(\zeta, \bar{z}) = \sum_{j=1}^{r} l_j \rho_j(\zeta, \bar{z}) \tag{47}$$

$$\rho_j(\zeta, \bar{z}) \equiv \hat{f}(\zeta) \eta_j \left(\eta_j \hat{f}(z)^\dagger \hat{f}(\zeta) \eta_j + I - \eta_j\right)^{-1} \eta_j \hat{f}(z)^\dagger.$$

The fundamental projection matrices ρ_j have the following properties, which can be obtained from the above expression, using (45):

$$\rho_j(\zeta, \bar{z})^\dagger = \rho_j(z, \bar{\zeta}) \qquad \rho_j(\zeta, \bar{z})^2 = \rho_j(\zeta, \bar{z}) \tag{48}$$

$$\text{tr}[\rho_j(\zeta, \bar{z})] = \text{tr}(\eta_j) \qquad \rho_j(0, 0) = \eta_j. \tag{49}$$

Thus $\rho_j(z, \bar{z})$ can be considered as η_j transported from the origin of $\mathcal{M}$ to an arbitrary point, as both matrices have the same set of eigenvalues. These matrices are sometimes also called momentum maps [16, 17], but we retain this term for their components, given by (46). Another explicit expression for ρ_j is derived in the appendix, namely

$$\rho_j(\zeta, \bar{z}) = \exp[-K^j(\zeta, \bar{z})] F_j Q_{F_j}(F_j) \qquad F_j \equiv \hat{f}(\zeta) \eta_j \hat{f}(z)^\dagger \tag{50}$$

where Q_F is a polynomial defined in the appendix, and its coefficients are also polynomials of $(\zeta, \bar{z})$. The degree of Q_F is $[\text{rank}(\eta) - 1]$, and $Q_{F_j} \equiv I$ if η_j has only one unit eigenvalue.

As follows from (33), the ρ-matrices are invariant, i.e.

$$\rho(g\zeta, \overline{gz}) = g\rho(\zeta, \bar{z})g^\dagger \qquad \forall g \in \mathcal{G} \tag{51}$$

which leads to the equivariance of the momentum maps, equation (25).

Thus a group representation is constructed in the space of holomorphic sections in $\mathcal{L}$, which according to the Borel–Weil–Bott theorem is the representation having the highest weight l. If some of the components of l are zero, the little group is essentially larger than the maximal torus $\mathcal{T}$, the form ω is degenerate, and neither the invariant volume nor the Poisson brackets can be introduced in $\mathcal{F}$. For such representations, the desired Kähler manifold is a section of the flag manifold, $\mathcal{M} \subset \mathcal{F}$. The contraction is considered in the next subsection.

4.4. The Kähler manifolds of lower ranks

Equations (24) show that ω is degenerate if there is a number of pairs (a, α), for which $\partial_\alpha T_a = 0$. Let us calculate the derivative using the representation (46). Applying the Lie derivative (16) to the definition (46), one obtains differential equations for $\rho(\zeta, \bar{z})$

$$D(\zeta)_\alpha \rho_j = \hat{e}_\alpha \rho_j - \rho_j \hat{e}_\alpha \rho_j \qquad \overline{D(z)_\alpha \rho_j} = \rho_j \hat{e}_\alpha - \rho_j \hat{e}_\alpha \rho_j . \qquad (52)$$

If $\hat{e}_\alpha$ commutes with ρ_j, the right-hand side vanishes. At the origin, $z = 0$, this is the case if $(\alpha \cdot w_j) = 0$ (cf equation (29)). Therefore if $(l \cdot \sigma) = 0$ for a number of root vectors $\sigma \in \Delta_s^+ \subset \Delta_g^+$, the Kähler (1,1)-form is degenerate at the origin, since the matrix $\omega_{\alpha\bar{\beta}}$ has a number of zero eigenvalues, as follows from (24). In this case, the (1,1)-form is degenerate everywhere in $\mathcal{F}$, because it is transported homogeneously from the origin. The set of the root vectors Δ_s^+ orthogonal to l is generated by a set of primitive roots γ_j for which $l_j = 0$. Correspondingly, the Lie algebra $s \subset g$ has the basis elements e_σ.

In other words, if $l_j = 0$ for some j, the Kähler structure on $\mathcal{F}$ is degenerate. Now $\mathcal{F}$ can be considered as a fibre bundle, $\mathcal{M}$ being its base, where the unitary group representation R_l generates a non-degenerate Kähler structure. The local coordinates on $\mathcal{M}$ are introduced by restriction $z_\sigma = 0$ for $\sigma \in \Delta_s^+$. Respectively, the little group of $\mathcal{M}$ is larger than the maximal torus

$$\mathcal{M} = \mathcal{G}/\mathcal{H} \qquad \mathcal{H} = \mathcal{S} \otimes T' \qquad (53)$$

where $\mathcal{S}$ is the semi-simple Lie group having s as its Lie algebra, and $T' \subset T$ is a torus generated by those basis elements h_j, for which $l_j \neq 0$. This construction has a clear interpretation in terms of Dynkin graphs [18]. Given a group representation R_l, one has to eliminate from the Dynkin graph the nodes for which $l_j \neq 0$. The number of such nodes is $k \leqslant r$, it may be called the *rank* of $\mathcal{M}$. The remaining nodes indicate a semi-simple Lie algebra $s \subset h \subset g$. As soon as the fundamental Kähler potentials are constructed for $\mathcal{F}$, the reduction to $\mathcal{M}$ is obtained by constraints $z_\sigma = 0$ in the definition of $f(z)$ in (32). The parabolic subgroup $\mathcal{P}$ is extended respectively; its Lie algebra is $p = g^- + t' + s^c$.

The representation dimensionality, given by the Weyl formula, can be represented also in terms of integrals on $\mathcal{M}$ (see equation (10))

$$N_l = \prod_{\alpha \in \Delta_g^+} \frac{(\alpha \cdot (l + \rho))}{(\alpha \cdot \rho)} = \frac{\int_\mathcal{M} d\mu(z, \bar{z})}{\int_\mathcal{M} \exp[-K(z, \bar{z})] d\mu(z, \bar{z})} \qquad (54)$$

where $\rho \equiv \sum_j w_j$. Actually, the product here is taken for $\alpha \in \Delta_g^+ \setminus \Delta_s^+$, since the weight vector l is orthogonal to all roots belonging to s.

5. Holomorphic cocycles and Killing vectors

The Lie derivative of elements of the parabolic subgroup $\mathcal{P}$, as given in (33), leads to a transport of the basis in the Lie algebra, which looks like a transformation of gauge fields

$$\hat{\theta}_a(z) \equiv D_a p(z; g)|_{g=e} = \hat{f}(z)^{-1} \tau_a \hat{f}(z) - \hat{f}(z)^{-1} \nabla_a \hat{f}(z) \quad \in p . \qquad (55)$$

The holomorphic cocycles in (18) and (22) are expressed in terms of these matrices in the fundamental representation of g

$$\varphi_a(z) = \sum_j l_j \operatorname{tr}(\eta_j \hat{\theta}_a(z)) . \qquad (56)$$

The cocycle condition (34) is equivalent to the following set of differential equations:

$$\nabla_a \hat{\theta}_b(z) - \nabla_b \hat{\theta}_a(z) + [\hat{\theta}_a(z), \hat{\theta}_b(z)] = f_{ab}^c \hat{\theta}_c(z) . \qquad (57)$$

The equalities (20) follow immediately. It is noteworthy that the left-hand side, like the gauge field strengths, is a curvature tensor on the manifold.

As soon as $\theta_a(z)$ belongs to the Lie algebra $p = g \setminus g^+$, the following equations hold ($\forall a$ and $\forall \beta \in \Delta_g^+$) which can be used to get explicit expressions for the Killing fields:

$$\mathrm{tr}\left(\hat{e}_\beta^\dagger \hat{\theta}_a(z)\right) = 0 = \mathrm{tr}\left(\hat{e}_\beta^\dagger \hat{f}(z)^{-1} \tau_a \hat{f}(z)\right) - \kappa_a^\alpha \, \mathrm{tr}\left(\hat{e}_\beta^\dagger \hat{f}(z)^{-1} \partial_\alpha \hat{f}(z)\right). \tag{58}$$

The Killing fields are obtained now in terms of the adjoint group representation $A(g)$. Let us use the following notation (capitals stand for the adjoint representation matrices):

$$Z \equiv z^\alpha E_\alpha \qquad A(z) = \exp(-Z) \qquad B(z) = (I - A(z))/Z. \tag{59}$$

Here E_α is the adjoint representation of $e_\alpha \in g^+$, so Z is a nilpotent (triangular) matrix, and the matrices $A(z)$ and $B(z)$ are polynomials in Z. The matrix $B(z)$ appears in the Cartan–Maurer one-form, which takes its values in the Lie algebra g

$$f(z)^{-1} \mathrm{d}f(z) = \mathrm{d}z^\alpha B_\alpha^a(z) \tau_a. \tag{60}$$

Thus the Killing vector fields κ_a^α satisfy the following set of linear equations:

$$\kappa_a^\alpha(z) B_\alpha^\beta(z) = A_a^\beta(z). \tag{61}$$

The solution exists, as soon as the minor of $B(z)$ corresponding to the Borel subalgebra does not vanish; $\kappa_a^\alpha(z)$ is a rational function of z.

6. Conclusion

The results presented here can be summarized as follows. For any unitary representation R_l of a compact simple group $\mathcal{G}$, one can construct a compact homogeneous Kähler manifold $\mathcal{M} \equiv \mathcal{G}/\mathcal{H}$ and a Hilbert space $\mathcal{L}$ of (locally) holomorphic functions which can be considered as a line bundle upon $\mathcal{M}$. The Lie algebra g is realized in $\mathcal{L}$ by means of linear differential operators, or by means of functions on $\mathcal{M}$, called symbols of g or momentum maps. If R_l is a non-degenerate representation, i.e. projections of the dominant weight vector l upon all primitive roots are positive integers, $\mathcal{H}$ is the maximal torus, $\mathcal{M}$ is the flag manifold, and its complex dimensionality equals the number of the positive roots of g. Otherwise, $\mathcal{M}$ is a section of the flag manifold and $\mathcal{H}$ is a torus times a simple group. The Kähler potential for $\mathcal{M}$ is constructed explicitly in terms of the fundamental group representation. It is given *as a superposition of a number of fundamental potentials with positive integer coefficients* (equation (42)).

The manifold $\mathcal{M}$ can be considered as a phase space of a dynamical system, and elements of the Hilbert space represent the generalized coherent states. The Poisson brackets are derived from the Kähler structure, providing a realization of the Lie algebra g in terms of the momentum maps. From this point of view, the construction described here is the Berezin quantization on homogeneous manifolds.

The Kähler structure can be introduced in $\mathcal{M}$ by means of (42) with arbitrary coefficients l_j, and one obtains a representation of the Lie algebra. Then the manifold can be still considered as a phase space of a dynamical system, but quantization (and the unitary group representation) is possible only for integer coefficients.

Extension of the present approach to non-compact groups and infinite-dimensional (*universal*) groups is the subject of a future work.

Acknowledgments

Appendix A. Derivation of equations (40) and (50)

We shall consider the fundamental Kähler potential, given in (38) according to [12], as a limit at $t \to 1$ of the following function:

$$K_t = \log \det[I - t\eta(I - M)\eta] \equiv \operatorname{tr} \log[I - t\eta(I - M)\eta] \qquad M \equiv \hat{f}^{\dagger} \hat{f}. \tag{A1}$$

Using the expansion in powers of t at $t < 1$, one obtains for $\eta = \eta^2$

$$K_t = \operatorname{tr} \{\log[I - t(I - F)] - (I - \eta)\log(1 - t)\} \qquad F \equiv \hat{f}\eta\hat{f}^{\dagger}. \tag{A2}$$

Let us introduce the following notation: N is the dimensionality of the matrix representation, $P_F(\lambda) \equiv \det(F - \lambda I)$ is the characteristic polynomial for matrix F, and $v = \operatorname{tr}(I - \eta)$ is the number of zero eigenvalues of η (and of F), $\tau = (1 - t)/t$. Now the result in (40) is evident from

$$K_t = \log \frac{P_F(-\tau)}{\tau^v} - (N - v)\log(1 + \tau) \qquad K \equiv \lim_{\tau \to 0} K_t = \log \det' F. \tag{A3}$$

(The function $\log \det'$, where $\det'$ is the Fredholm determinant of an elliptic operator in the Hilbert space with its zero modes excluded, also appears as the derivative of the operator ζ-function at zero, cf, for example, [19].)

Similarly, one obtains equation (49) for the matrix ρ, which can be represented by the same limit, namely

$$\rho = \lim_{\tau \to 0} F(F + \tau I)^{-1}. \tag{A4}$$

In other words, the eigenvalues of ρ vanish together with the eigenvalues of F, and are equal to 1 otherwise. As soon as the rank of F equals the rank of η, one can write the characteristic polynomial as follows:

$$P_F(\lambda) = (-\lambda)^v [D_F - \lambda Q_F(\lambda)] \qquad D_F \equiv \det' F. \tag{A5}$$

Here $Q_F(\lambda)$ is a polynomial of degree $N_1 = N - v - 1$, and its coefficients are polynomials in $(\zeta, \bar{z})$, which are expressed in terms of the traces $f_n = \operatorname{tr}(F^n)$ for $n = 1, \ldots, N_1$, namely

$$Q_F(\lambda) = (-\lambda)^{N_1} + f_1(-\lambda)^{N_1 - 1} + \tfrac{1}{2}(f_1^2 - f_2)(-\lambda)^{N_1 - 2} + \cdots. \tag{A6}$$

The result in (50) follows, as soon as $D_F = \exp(K)$, which is also a polynomial in f_n.

Appendix B. Example: Unitary groups and the Grassmann manifolds

For $\mathcal{G} = SU(N)$, the fundamental group representation is N-dimensional, and the index in (42) is in the interval $1 \leqslant j \leqslant r = N - 1$. Up to a normalization factor, the eigenvalues of $i\hat{h}_j$ are -1 (j times), j, and 0 ($(N - j - 1)$ times). The corresponding projection matrix η_j has j values 1, other $(N - j)$ values are 0. The local coordinates corresponding to the positive roots are elements of a triangular matrix $\hat{z}$, i.e. z_{jk}, $1 \leqslant j < k \leqslant N$ (other elements of the matrix are zero), and the complex dimensionality of $\mathcal{F}$ is $\tfrac{1}{2}N(N - 1)$. The matrix $\hat{f}(z)$ is triangular; its diagonal elements are 1, and polynomials in z^{α} stand above the diagonal. The manifold $\mathcal{F}$ has an additional symmetry under a reflection of the root

space, so that $j \to N - j + 1$, and $\eta_{N-j+1} \to \sigma(I - \eta_j)\sigma^{-1}$, where σ is a matrix reversing the order of components. Matrices $F_j = \hat{f}\eta_j\hat{f}^\dagger$ can easily be obtained in a general form.

The Grassmann manifold $\mathcal{M} = \mathrm{Gr}\,(p, q) \equiv U(p + q)/U(p) \otimes U(q)$ (where $1 \leqslant q \leqslant p < N \equiv p + q$) is a rank-one section of $\mathcal{F}$. The group representation realized in $\mathcal{M}$ is specified with a positive integer l, and $l_j = l\delta_{jq}$. The complex dimensionality of the manifold is pq, and the local coordinates are elements of a $p \times q$ matrix $\hat{z}$, so that the elements of $\hat{f}$ are $f_{jk} = \delta_{jk} + z_{j,k-p}$, where $1 \leqslant j \leqslant p$, and $p + 1 \leqslant k \leqslant p + q$. The resulting Kähler potential is

$$K(z, \bar{z}) = l \log \det(I_q + \hat{z}^\dagger \hat{z}). \tag{B1}$$

For $q = 1$, $\hat{z}$ is a complex vector, $\mathcal{M} \equiv \mathbb{CP}^{N-1}$ is the complex projective space, and the Fubini–Study form [13] appears from the Kähler potential. The metrics are 'quantized', since l is an integer.

It is noteworthy that for any compact Lie group the Kähler structure can be obtained by restriction from a unitary group, as soon as the group is embedded in it, $\mathcal{G} \subset SU(N)$.

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